DESCRIBING MODEL OF THE TOPOLOGICAL RELATION IN SPHERICAL SURFACE QUATERNARY TRIANGULAR MESH

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Commission II,ICWG-II-IV

KEY WORDS: Spherical surface QTM, Euler_number, Topological relation

ABSTRACT:

Spherical surface QTM (Quaternary Triangular Mesh) is one of an efficient tool to deal with the global data because of its advantages of multi-resolution and hierarchy. Based on characters of spherical surface QTM, set muti-operators and Euler_number of symmetric difference will be presented to describing and computing model of the topological relation in spherical surface QTM. In this model, the topological invariant (empty or not empty) of the result of the intersection, difference, difference by and symmetric difference between two spatial objects are used to partially distinguish their five traditional topological relation, and then the Euler number of the result of symmetric difference between two spatial objects is introduced to confirm the other three topological relations of disconnected / disjoin, contain / overlap, and contained by / overlaped by which the traditional methods can't distinguish.

1. INTRODUCTION

Up to now, the researches about the global data are almost based on map projection, i.e. the real world (3-dimension) is transformed into 2-dimension planar. As a result, the global data are inevitably produced overlaps and gaps. The map projection transforms the sphere manifold to planar Euclidean space, therefore, the distance, orientation and area in large field are not accurate at all (Hu, 2001). In addition, people just use planar map after map projection to browse and analyze, then give conclusions and make decisions. If large quantities of global information can be directly dealt with on the spherical surface in a computer, the map projection, which involves complex calculations, can be avoided. With the development of computer technique, it's possible to directly storage, manage and analyze the large quantities of global information. Making the global dada directly on the sphere and constructing sphere dynamic data model is one of the key problems to the Digital Earth (Zhao et al., 2002).

Quaternary Triangle Mesh on spherical surface (QTM on spherical surface) is a partition of the Earth's surface, which usually consists of increasingly finer resolution grids. It has the advantages of hierarchical structure, continuous ordering and equivalent subdivision et al (Nulty, 1993; Bartholdi and Goldsman, 2001). Therefore, QTM on spherical surface is one of the efficient methods to deal with global data rupture, distortion and topological difference caused by the planar raster model. Several works have been done about it in many applications such as global hierarchical structure organization of spatial data generalization (Dutton 1989, 1997), global DEM data compression (), global environment sampling (White et al. 1992), global continuous indexing (Bartholdi and Goldsman 2001) and the generation of spherical Voronoi diagram (Chen et al.).

However, research achievements about QTM on spherical surface are confined to design of the system, inter-operation, error-distribution, coding and several applications, et al. Few can be found about its topological model. Descartes plane is isotropic, which has the perfect linear structure, so there exists Descartes coordinates system to be applicable for every point in Descartes plane. However, spherical surface is only an anisotropic manifold but not an isotropic Euclidean space. That is to say, no single coordinates system can be set up to express every point in spherical surface. Although longitude and latitude coordinate system can be constructed to describe all points in spherical surface, South Pole and North Pole are exceptions. The latitude of North Pole is north 90 degree, but the longitude of it is not sure; the same is as South Pole. Spherical surface as 2D manifold is not the homomorphism of 2D Descartes plane and its subsets, therefore, there must be paradox and error if the topological model of raster space is directly used for spherical surface.

This paper is concerned with basic topology model for QTM space on spherical surface. Digital topology provides a sound mathematical basis for various image-processing applications including surface detection, border tracking, and thinning in 2D Euclidean space (Kong 1986; Rosenfield 1975). We often use voxel representation to describe objects on a computer. Specifically, QTM space on spherical surface is partitioned into unit triangles. In this representation, an object in spherical surface is described by an array of bits. In this way, an object on spherical surface can be defined as an array augmented by a neighbourhood structure. The emphasis of this paper is on the differences between planar and spherical surface. It is specific to the basic topology model on the surface of an earth, and thus, the ellipsoidal nature of the earth and its vertical dimension are not considered.

The paper is organized as follows. Next section presents the definitions of QTM space on spherical surface based on manifold. In Section 3, the basic topology model of QTM space on spherical surface is discussed. Section 4 will present a new model to express and compute the spherical surface raster topological relation. In the end, the discussions and the future works are given.

2. THE DEFINITION OF QTM SPACE ON SPHERICAL SURFACE BASED ON MANIFOLD

Regular grid sampling structures in the plane are a common spatial framework for many applications. Constructing grids with desirable properties such as equality of area and shape is more difficult on a spherical surface (White et al. 1998). To deal with the problems on the Earth conveniently, it is necessary to construct a similar regular mesh structure as a common spatial framework for spherical surface just as planar. Therefore, it is necessary to subdivide the spherical surface according to its characteristics. There are three steps to get the QTM space on spherical surface.

2.1 Initial partition of the spherical surface

The Platonic solids are reasonable starting points for a spherical surface subdivision (shown in Figure 1). Three of the five polyhedrons have triangular faces, such as the tetrahedron (four faces), the octahedron (eight faces), and the icosahedron (20 faces). The other Platonic solids are the cube (six faces) and the pentagonal dodecahedron (12 faces). The icosahedron has the greatest number of initial faces, and would therefore show the least distortion in the subdivision. However, the larger number of faces makes it somewhat harder to deal with the problems through the borders of the initial faces. In a word, the spherical surface is more easily covered by triangles, and the triangles of the initial partition need not be equilateral. Distortion could be decreased considerably by dividing each equilateral triangular side of an initial Platonic figure into equivalent scalene triangles (White et al. 1998).

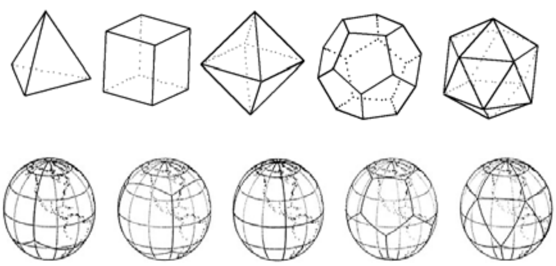


Figure 1. Platonic solids and their spherical surface subdivision (White et al. 1992)

The octahedron has more distortion, but it has the advantage that its faces and vertices map to the important global features: meridians, the equator, and the poles (Goodchild and Shiren 1992). Therefore, in this paper octahedron is selected as common initial partition in which eight base triangles are produced.

2.2 Subdivision of triangular cells

There are several ways to hierarchically subdivide an equilateral triangle such as quaternary subdivision and binary subdivision (shown in Figure 2). All of these are subject to distortion when transferred to the spherical surface. Different decisions will have different effects on the uniformity of shape and size of cells within a given level of the hierarchy, as well as on the ease of calculation. Here, the quaternary subdivision is selected, in which a triangle is subdivided by joining the midpoints of each side with a new edge, to create four sub-triangles.

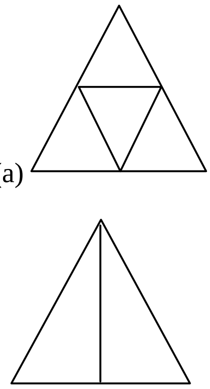


Figure 2. Quaternary subdivision (a) and binary subdivision (b)

The quaternary subdivision is a good compromise. It is relatively easy to work with, and non-distorting on the plane: a planar equilateral triangle is divided into four equilateral triangles. But a spherical base triangle may be divided into four equivalent triangles. The result of subdivision based on octahedron with quaternary subdivision is as follows in Figure 3 (Dutton 1996).

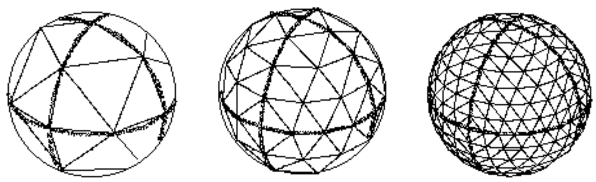


Figure 3. The result of subdivision based on octahedron (Dutton 1996)

2.3 The definition of QTM space on spherical surface based on manifold

Manifold is the extension of Euclidean just because every point in manifold has a homeomorphism of an open set in Euclidean. So local coordinates system can be set up for every point in manifold. It seems that manifold is a result plastered with many Euclidean spaces. It can be proved that spherical surface is a 2-dimension smooth manifold (Evidence omitted). If the spherical surface is divided by quaternary subdivision based on octahedron, the QTM space on spherical surface is $8 \times 4^N \ (N = \{0,1,...,n-1\})$ regular mesh based on finite discrete space, expressed as T^2. In the first level, spherical surface has the 8 base triangles, which are local coordinates systems of manifold. The relationship between 8 local coordinate systems can be described by spherical surface space filling curves (shown as Figure 4), which is a continuous mapping from a one-dimensional interval, to the points on the spherical surface.

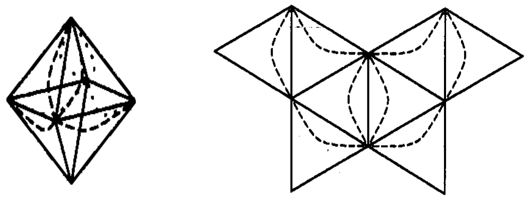
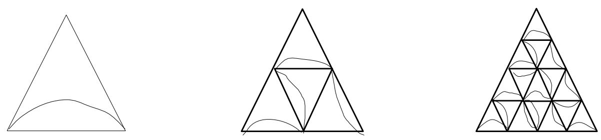


Figure 4. Spherical space filling curves based on octahedron with quaternary subdivision

To every base triangle, quaternary spherical surface space filling curve still can be used to express the relationship between every sub-triangle. In quaternary subdivision, the relationship between sub-triangles can be depicted with quaternary spherical surface space filling curve (shown as Figure 6). In given resolution, QTM space on spherical surface can be continuously indexed by quaternary spherical space filling curve (Details in Bartholdi 2001). Comparing with the other model (Dutton 1991), spherical surface digital space has the advantage of continuous ordering. It makes us to index QTM space on spherical surface continuously to allow quick and efficient search at multi-scale. At the same time, QTM space on spherical surface has the intrinsic disadvantage that the triangle is equivalent but not equal with each other.



(a)level (b)level2 (c)level3

Figure 6. The quaternary spherical surface space-filling curve

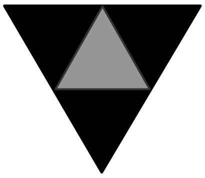
Planar digital space is a simple Euclidean space, but spherical surface digital space is a more complex manifold. So spherical surface digital space is not the simple copy of planar digital space. It has some special properties just as follows. Spherical surface digital space is not a Euclidean space, that is to say, it is no homomorphous to planar and no single coordinates system can be set up to express every point in spherical surface. Although cells of spherical surface digital space are approximately equivalent, it still has a multi-scale and continuous ordering advantages (Bartholdi and Goldsman 2001).

3. THE BASIC TOPOLOGY MODEL OF SPHERICAL SURFACE DIGITAL SPACE

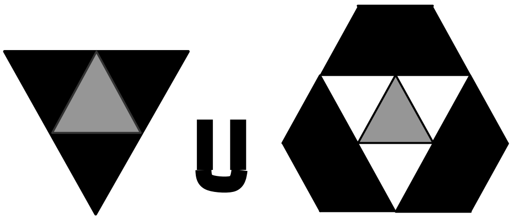
From the definition of spherical surface digital space, T^2 is the result of partitioning the connected spherical surface into small triangular pieces that cover the whole spherical surface space. Each spherical triangle is viewed as an element, called "spel" (short for spatial element). All the spels in the spherical surface can form a new set, which can be named as grid set T^2. The set T^2 can then be regarded as the hardware of the spherical surface digital space. The transitive closure δ of the adjacency relation between the two spels in T^2 can be considered as software. This system can be expressed as $\langle T^2, \delta \rangle$, where δ is the binary relations. This binary relation determines the connectedness between the spels in T^2. $\langle T^2, \delta \rangle$ is also referred to as "spherical surface digital topology". S^2 is a connected space, but the T^2 is not connected space. In T^2, this implicit assumption of connectedness in S^2 no longer works.

3.1 General definitions and notations

Points of T^2 associated with triangles that have value 1 are called black points, and those associated with triangles with value 0 are called white points. The set of black points normally corresponds to an object in the digital image. First, we consider objects as subsets of the spherical surface digital space T^2. Elements of T^2 are called "spels" (short for spatial element). The set of spels which do not belong to an object O is included in T^2 constitute the complement of the object and is denoted by $\overline{O}$. Any spel can be seen as a unit triangle centered at a point with integer coordinates. Now, we can define some binary symmetric antireflexive relations between spels. Two spels are considered as 3-adjacency if they share an edge and 12-adjacent if they share a vertex. For topological considerations, we must always use two different adjacency relations for an object and its complement (shown as Figure 7). We sum this up by the use of a couple (n, n') with $(n, n') = \{3;12\}$, the $\mathrm{n\text{-}adjacency}$ being used for the object and the $\mathrm{n'\text{-}adjacency}$ for its complement. By transitive closure of these adjacency relations, we can define another one: connectivity between spels. We define an $\mathrm{n\text{-}path}\ \pi$ with a length k from spel a to spel b in O included in T^2 as a sequence of $\mathrm{voxels\,(i)\ i = 0; \ldots, k}$, such that for $0 \le \mathrm{i} \le \mathrm{k}$, the spel v_i is $\mathrm{n\text{-}adjacent}$ or equal to v_{i+1}, with $\mathrm{v}_0 = \mathrm{a}$ and $\mathrm{v}_\mathrm{k} = \mathrm{b}$. Now we define connectivity: two voxels a and b are called $\mathrm{n\text{-}connected}$ in an object O if there exists an $\mathrm{n\text{-}path}\ \pi$ from a to b in O. This is an equivalence relation between spels of O, and the $\mathrm{n\text{-}connected}$ components of an object O are equivalence classes of spels according to this relation. Using this equivalence relation on the complement of an object we can define a background component of O as an $\mathrm{n'\text{-}connected}$ component of O'.



3-adjacent



(b) 12-adjacent

Figure 7. The definition of 3-adjacent and 12-adjacent

In 2D spherical surface digital space, we consider spherical surface triangle mesh to express spherical surface digital image. In this paper, points refer to grid points in spherical surface digital space unless stated otherwise. Two nonempty sets of points S_1 and S_2 are said to be 3-adjacent or 12-adjacent if at least one point of S_1 is 3-adjacent or 12-adjacent to at least one point of S_2. The adjacency definition is important not only in the computation of raster distance between two spels but also in topological analysis (LI et al. 2000). Let S be a nonempty set of points. An 3-path between two points p, q in S means a sequence of distinct points $p = p_0, p_1, ..., p_n = q$ of S such that p_i is 3-adjacent to p_{i+1}, $0 \le i < n$. Two points $p, q \in S$ are 3-connected in S if there exists an 3-path from p to q in S. An 3-component of S is a maximal subset of S where each pair of points is 3-connected.

A 2D spherical surface digital object E can be defined as the set of black points that is spatial entity in spherical digital space. Samely, $T^2 - E$ is the set of white points, which is called the background of E. 3-adjacency or 12-adjacency are the adjacencies used for finding 3-components and 12-components in E and $I - E$ respectively. In this paper, we use 12-adjacency for black points and 3-adjacency for white points and call 12-components of E black components and 3-components of $I - E$ white component. The basic topological components of a spatial entity in spherical surface digital space are still interior, boundary and exterior. A point $p \in E$ is called an interior point of E if $N(p) \subset E$, otherwise p is called a border point of E. The set of all interior points of E is called the interior of E and is denoted as E°. The set of all border points of E is called the border of E and is denoted as ∂E. The closure of E is denoted as $\overline{E}$. The relationship between interior, closure and boundary is as follows:

$$\mathrm{E}^o \bigcap \partial \mathrm{E} = \Phi$$

$$\mathrm{E}^o \bigcup \partial \mathrm{E} = \overline{\mathrm{E}}$$

$$\partial \mathrm{E} = \overline{\mathrm{E}} \bigcap (\overline{\mathrm{E}})^-$$

3.2 Topological paradox associated with definition of adjacency in T2

The classical Jordan curve theorem says that the complement of a Jordan curve in the Euclidean plan R^2 consists of exactly two connectivity components. This theorem is the basic topological property in vector space and it would be preferable to keep it in the Z^2 raster space. So a topological paradox in Z^2 has arisen (figure 8). Kong and Rosenfeld has solved this problem in Z^2 if the white spels are defined as being 4-connected and black spels being 8-connected, or vice versa.

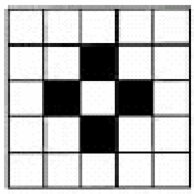


Figure 8. Topological paradox in raster space (from LI et al. 2000)

However, no one has discussed the topological paradox in T^2. In Figure 9, there are six black spels, one gray spel and some white spels. The gray spel is surrounded by the six black spels. If 12-adjacency is defined, the black spels are connected and should form a closed line; however, this black line cannot separate the central gray spel from the white spels. If 3-adjacency is defined, the black spels do separate the central gray spel from the white spels; however, these black spels are totally disconnected and thus no closed line has been formed by the black spels in this case. So this leads to the topological paradox in raster space T^2. To deal with this paradox, the white spels are defined as being 3-connected and black spels 12-connected, vice versa. In spherical surface digital space, background and object have the different connectedness. That is to say, the spatial entity in spherical surface is defined as being 12-connected, but the background is defined as being 3-connected. So, the six black spels defined as 12-connected should be connected. However, gray spel and black spels just as background should be not connected if the background is defined as being 3-connected. So the continuous curve (connected path in T^2) separate the spherical surface two parts.

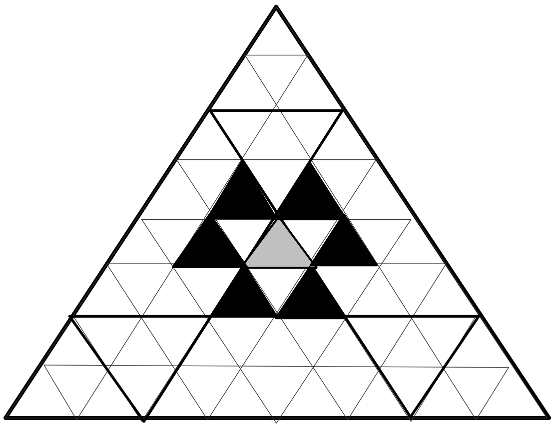


Figure 9. Topological paradox in raster space T^2

But why this topological paradox happens? We use the six spels in figure 10 to explain it. In reality, when one considers spels 1, 3 and 5 to be connected, one has already implicitly assumed that P belongs to the black line. On the other hand, when one considers spels 2, 4 and 6 to be connected, one has already implicitly assumed that P belongs to the white spels. That is, the point P belongs to two different things (LI et al. 2000). If the black spels represent spatial entities and the white spels represent the background, then point P belongs to both the background and the entity at the same time, thus having dual meanings. This of course leads to paradox–a kind of ambiguity. To solve the problem, one must eliminate the dual meanings of point P. One should only allow P to belong to either the entity or the background but not both. In this paper, the spels belonging to background are defined as 3-connected, however, the spels belonging to the object are defined as 12-connected.

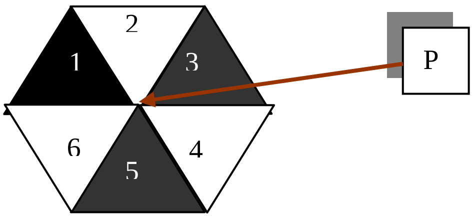
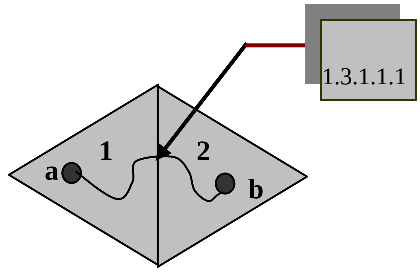


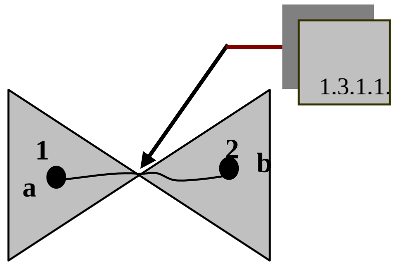
Figure 10. Topological paradox caused by the ambiguity at point P

3.3 Relationship between topology between spherical surface digital space and spherical surface continuous space

The connectedness of raster space is based on the adjacency of two neighboring spels (LI et al. 2000). In spherical surface digital space, there is a common line (Figure 11a) between the two spels in the case of 3-connectedness. On the other hand, in the case of 12-connectedness, the common part could be either a line, a point, or both. In other words, there is at least a point in common if the two spels are to be connected. If an arbitrary (vector) point is selected from each spel, say ``a'' and ``b'', then the path from ``a'' to ``b'' intersects the common line at P. Points ``a'', ``b'' and P are points in vector space. Points ``a'' and P are connected in the left spel and points P and ``b'' are also connected in the right spel in vector space. As the connectedness is transitive, points ``a'' and ``b'' are therefore connected. As a result, any point in the left spel is connected to any point in the right spel. It means that the connectedness concept in vector space has been implicitly adopted when the connectedness concept in raster space is discussed.



(a) in the case of 3-adjacency



(b) in the case of 12-adjacency

Figure 11. Implicit dependency of topological connectedness in T^2

4. CONCLUSION

SGDM (Sphere Grid Data Model) is an efficient method to deal with the global data because of the advantages of multi-resolution and hierarchy. However, SGDM has no distinct descriptions and lack of round mathematical basis for various applications. This paper gave the definition of spherical surface digital space, which has the characters as follows:

- Similar regular grids based on spherical surface discrete space.
- Spherical spacefilling curves can be used to express the relationship between basic local coordination.
- No single coordination system can express every point in the spherical surface.
- Multi-scale and continuous ordering.

As another important part, this paper set up the basic topology model which include the topological structure of spherical surface digital space, the basic topological components of a spatial entity in T^2, topological paradox associated with definition of adjacency in T^2 and so on. This paper is just an introduction to studying the characterization of 2D digital spherical manifold and the Jordan separation theorem, which are all round mathematic basis of spherical spatial computing and reasoning.

ACKNOWLEDGEMENTS

The work presented in this paper was substantially supported by an Outstanding Youth Award from the Natural Science Foundation of China (under grant No.40025101). I also thank my two supervisors, Prof. Chen Jun and Dr. Zhao Xuesheng, whose comments allow me to improve this article.

REFERENCES

Bartholdi. III and Goldsman P., 2001. Continuous indexing of hierarchical subdivisions of the globe, Int. J. Geographical Information Science, 15(6): 489-522.

Bieri H. and Nef W., 1984, Algorithms for the Euler characteristic and related additive functional of digital objects. Computer Vision Graphic Image Process. 28: 166–175.

Chen J., Zhao X., and Li Z., 2003. Algorithm for the generation of Voronoi diagrams on the sphere based on QTM . Photogrammetric Engineering or Remote Sensing, 69(1): 79-90.

Cohn A., Bennett B., Goodday J. and Gotts N., 1997. Qualitative Spatial Representation and Reasoning with the Region Connection Calculus. Geoinformatica, 1(3): 1-44.

Dutton G., 1991. Polyhedral hierarchical tessellations: The shape of GIS to come.. Geographical Information Systems, 1(3): 49-55.

Dyer C. R., 1980. Computing the Euler number of an image from its quadtree. Computer Graphics Image Process. 13, 270–276.

Egenhofer M., Sharma J., and David M., 1993. A critical comparison of the 4-intersection and 9-intersection models for spatial relations: formal analysis. Auto-Carto, 11:1-11.

Frank A. U., 1992. Spatial Concepts, Geometric Data Model and Geometric Data Structures. Computers & Geosciences, 18: 409-417.

Goodchild M.F. and Yang Shiren, 1992. A Hierarchical Data Structure for Global Geographic Information Systems, Computer Vision and Geographic Image Processing, 54(1): 31-44.

Kong T.Y., 1986. Digital topology with applications to image processing. Doctoral dissertation, University of Oxford.

LI Z.L., LI Y.L, CHEN Y.Q., 2000. Basic Topological Models for Spatial Entities in 3-Dimensional Space. GeoInformatica, 4(4): 419-433.

Malandain G., Bertrand G., and Ayache N., 1993. Topological segmentation of discrete surfaces. Int. J. Comp, 10(2): 183–197.

Rosenfield A., 1975. A characterization of parallel thinning algorithms, Inform. Control., 29: 286–291.

Rosenfield A., 1979. Digital Topology. Am. Math. Month, 86: 621-630.

Raskin, G. R., and Fellow, V., 1994. Spatial Analysis on the Sphere: A Review. NCGIA report, 94-7, 23pp.

Saha P. K., 1996. Chaudhuri B. B., 3D Digital Topology under Binary Transformation with Applications. Computer vision and image understanding, 63(3): 418-429.

White D., Kimmerling J. and Overton W.S., 1992. Cartographic and Geometric Components of a Global Sampling Design For Environment Monitoring, Cartography & Geographical Information Systems, 19(1): 5-22.

White, D., Kimberling, A. J. and Song L. 1998. Comparing area and shape distortion on polyhedral-based recursive partitions of the sphere. International Journal of Geographical Information Systems, 12(8): 805-827.

Winter S., 1999. Topological Relations in Hierarchical Partitions. Spatial Information Theory, COSIT'99, Springer, 141-156.

Zhao X.S., Chen J., and Li Z.L. 2002. A QTM-based algorithm for generation of the voronoi diagram on a sphere. Advances in Spatial Data Handling, Published by Springer, Berlin, 269-285.

Zhao X.S., 2002, Spherical Voronoi data model based on QTM. *Ph.D. thesis*, China University of Mining technology (Beijing), Beijing, 105pp.