ESTIMATION OF MODEL ERROR OF LINE OBJECTS

Wenzhong Shi[a] ; Sio-Kei Cheong [b]; Eryong Liu[a, c]

[a]Department of Land Surveying and Geo-Informatics,The Hong Kong Polytechnic University, Hong Kong

[b]Department of Land Management, Renmin University of China, China

[c]School of Environment and Spatial Informatics, China University of Mining and Technology, Xuzhou, China

Commission II

KEY WORDS: model error, truncation error, lines, positional error, geographic information science

ABSTRACT:

This paper presents a method for estimating the model error, specifically, a truncation error, of lines. The generic method for describing truncation error for $n-$ dimensional spatial features is developed first based on the numerical analysis theory. The methods for describing truncation error of the lines in three-dimensional and two-dimensional space, which are the two frequently used cases, are then derived as special cases of the generic method. This research is a further development of earlier work on line error modelling, which were mainly focusing on the propagated error rather than model error.

1. INTRODUCTION

Line feature error is one of the fundamental issues in the areas of modeling uncertainties in spatial data and spatial data quality control. Placing unrealistically high trust in the accuracy of data in GIS may mislead and seriously inconvenience GIS users.

Normally error of lines can be classified as the following categories: (a) model error which is related to the interpolation models for a line, and (b) propagated error which is related to the error of the original nodes that composite the line. The first type error can, in many cases, become a dominant part of the overall line feature error. The second type of error has been widely researched in the past, while study on model error of lines in GIS are relatively less reported. This research intend to investigate the model error of the lines, as a further supplement to the realy studies on propagated error of lines.

Straight line features in GIS include line segments, polylines, polygons, with the latter being regarded as a special representation of a polyline, in which the start point is identical to the end point. Line segments are also a special representation of a polyline – the number of the component line segments being equal to one. Hence, in this study, a polyline is taken as the representative feature for straight line feature error. A third-order curve is also taken as an examplefor truncation error modeling.

2 METHOD FOR DESCRIBING TRUNCATION ERROR OF STRAIGHT LINE FEATURES IN AN $n-$ DIMENSIONAL SPACE

Straight line features in geographic information science can include line segments, polylines, and polygons. The polyline is taken as a generic form of a straight line feature in the following discussion.

2.1 Definition of a polyline in an $n-$ dimensional space

A polyline in $n-$ dimensional space is composed of vectors: $Q_{n1}=[x_{11},x_{12},\cdots,x_{1n}]^T$, $Q_{n2}=[x_{21},x_{22},\cdots,x_{2n}]^T,\cdots$, $Q_{ni}=[x_{i1},x_{i2},\cdots,x_{in}]^T,\cdots$, $Q_{nm}=[x_{m1},x_{m2},\cdots,x_{mn}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points (as shown in Figure 1). The polyline is represented by the line segments $Q_{n1}Q_{n2},Q_{n2}Q_{n3},\cdots,$ $Q_{ni}Q_{n,i+1},\cdots,Q_{n,m-1}Q_{nm}$.

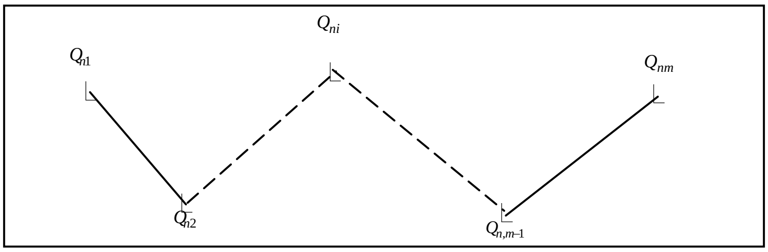

Figure 1. An example of a polyline in $n-$ dimensional space

Any point on the line segment $Q_{ni}Q_{n,i+1}(i=1,2,\cdots,m-1)$ is then represented by

$$Q_{nir} = \begin{bmatrix} x_{i1r} \\ x_{i2r} \\ \vdots \\ x_{inr} \end{bmatrix} = (1-r)\,Q_{ni} + rQ_{n,i+1}$$

$$= \begin{bmatrix} (1-r)x_{i1}+rx_{i+1,1} \\ (1-r)x_{i2}+rx_{i+1,2} \\ \vdots \\ (1-r)x_{in}+rx_{i+1,n} \end{bmatrix} \triangleq \begin{bmatrix} p_{i1}(r) \\ p_{i2}(r) \\ \vdots \\ p_{in}(r) \end{bmatrix} \tag{1}$$

$$\left(i=1,2,\cdots,m-1,\quad r\in[0,1]\right)$$

The true polyline of $Q_{n1}Q_{n2}\cdots Q_{ni}\cdots Q_{nm}$ can be represented by

$$\phi_{nir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \\ \vdots \\ \mu_{inr} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \\ \vdots \\ f_{in}(r) \end{bmatrix} \left(i=1,2,\cdots,m-1,\quad r\in[0,1]\right) \tag{2}$$

2.2 The truncation polyline error in an $n-$ dimensional space

Some assumptions about the functions $f_{ij}(r)\,(i=1,2,\cdots,m;$ $j=1,2,\cdots,n;r\in[0,1])$ need to be made in order to estimate interpolation error – a model error.

It is assumed that $f_{ij}(r)\in C^2[0,1]$, the following is then obtained

$$\left|R_{ij}(r)\right| = \left|f_{ij}(r)-p_{ij}(r)\right| \le \frac{1}{2} M_{2ij}\, r\,(1-r)$$
$$\le \frac{1}{8} M_{2ij} \triangleq T_{ij} \tag{3}$$

Where $M_{2ij} = \max\limits_{r\in[0,1]}\left|f''_{ij}(r)\right|$.

Hence, the polyline truncation error can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \\ \vdots \\ T_{in} \end{bmatrix} (i=1,2,\cdots,m-1).$$

Next, truncation error models for straight line features in three-, two- dimensional spaces are derived based on the generic truncation error model for straight line features in n-dimensional space. These two truncation error model cases may be frequently used in real world geographic information systems.

3 THE TRUNCATION ERROR MODEL FOR STRAIGHT LINE FEATURES IN THREE-DIMENSIONAL SPACE

3.1 Definition of a polyline in three-dimensional space

A polyline in three-dimensional space is composed of vectors: $Q_{31}=[x_{11},x_{12},x_{13}]^T$, $Q_{32}=[x_{21},x_{22},x_{23}]^T,\cdots,$ $Q_{3i}=[x_{i1},x_{i2},x_{i3}]^T,\cdots,$ $Q_{3m}=[x_{m1},x_{m2},x_{m3}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points.

Any point on the line segment $Q_{3i}Q_{3,i+1}(i=1,2,\cdots,m-1)$ is then represented by

$$Q_{3ir} = \begin{bmatrix} x_{i1r} \\ x_{i2r} \\ x_{i3r} \end{bmatrix} = (1-r)\,Q_{3i} + rQ_{3,i+1}$$

$$= \begin{bmatrix} (1-r)x_{i1}+rx_{i+1,1} \\ (1-r)x_{i2}+rx_{i+1,2} \\ (1-r)x_{i3}+rx_{i+1,3} \end{bmatrix} \triangleq \begin{bmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{bmatrix} \tag{4}$$

$$\left(i=1,2,\cdots,m-1,\quad r\in[0,1]\right)$$

The true polyline of $Q_{31}Q_{32}\cdots Q_{3i}\cdots Q_{3m}$ can be represented by

$$\phi_{3ir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \\ \mu_{i3r} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \\ f_{i3}(r) \end{bmatrix} \left(i=1,2,\cdots,m-1,\quad r\in[0,1]\right) \tag{5}$$

3.2 The truncation error of a polyline in three-dimensional space

Some assumptions about the functions $f_{ij}(r)(i=1,2,\cdots,m;$ $j=1,2,3;r\in[0,1])$ need to be made, in order to enable the estimation of the interpolation error.

It is assumed that $f_{ij}(r) \in C^2[0,1]$, giving

$$\begin{aligned}&\left|R_{ij}(r)\right| = \left|f_{ij}(r) - p_{ij}(r)\right| \le \frac{1}{2} M_{2ij} r(1-r) \\ &\le \frac{1}{8} M_{2ij} \triangleq T_{ij}\end{aligned} \tag{6}$$

where $M_{2ij} = \max\limits_{r\in[0,1]} \left|f''_{ij}(r)\right|$.

Hence, the polyline truncation error can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{bmatrix} (i=1,2,\cdots,m-1).$$

4 THE TRUNCATION ERROR MODEL FOR STRAIGHT LINE FEATURES IN TWO-DIMENSIONAL SPACE

4.1 Definition of a polyline in two-dimensional space

A polyline in two-dimensional space is composed of vectors: $Q_{21} = [x_{11}, x_{12}]^T$, $Q_{22} = [x_{21}, x_{22}]^T, \cdots$, $Q_{2i} = [x_{i1}, x_{i2}]^T, \cdots$, $Q_{2m} = [x_{m1}, x_{m2}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points.

Any point on the line segment $Q_{2i}Q_{2,i+1} (i=1,2,\cdots,m-1)$ is then represented by

$$\begin{aligned}Q_{2ir} &= \begin{bmatrix} x_{i1r} \\ x_{i2r} \end{bmatrix} = (1-r)\,Q_{2i} + rQ_{2,i+1} \\ &= \begin{bmatrix} (1-r)x_{i1} + rx_{i+1,1} \\ (1-r)x_{i2} + rx_{i+1,2} \end{bmatrix} \triangleq \begin{bmatrix} p_{i1}(r) \\ p_{i2}(r) \end{bmatrix} \\ &\left(i=1,2,\cdots,m-1,\ \ r\in[0,1]\right)\end{aligned} \tag{7}$$

The true polyline of $Q_{21}Q_{22}\cdots Q_{2i}\cdots Q_{2m}$ can be represented by

$$\phi_{2ir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \end{bmatrix} \left(i=1,2,\cdots,m-1,\ \ r\in[0,1]\right) \tag{8}$$

4.2 The truncation error of a polyline in two-dimensional space

Some assumptions about the functions $f_{ij}(r) (i=1,2,\cdots,m;$ $j = 1, 2; r \in [0,1])$ need to be made in order to be able to estimate the interpolation error.

It is assumed $f_{ij}(r) \in C^2[0,1]$, giving:

$$\begin{aligned}&\left|R_{ij}(r)\right| = \left|f_{ij}(r) - p_{ij}(r)\right| \le \frac{1}{2} M_{2ij} r(1-r) \\ &\le \frac{1}{8} M_{2ij} \triangleq T_{ij}\end{aligned} \tag{9}$$

Where $M_{2ij} = \max\limits_{r\in[0,1]} \left|f''_{ij}(r)\right|$.

Hence, the polyline truncation error can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \end{bmatrix} (i=1,2,\cdots,m-1).$$

5 THE TRUNCATION ERROR MODEL FOR A CURVE LINE IN AN $n-$ DIMENSIONAL SPACE

5.1 Define a curve line in an $n-$ dimensional space

The curve line is formed by the n-dimensional vectors: $Q_{n1} = [x_{11}, x_{12}, \cdots, x_{1n}]^T$, $Q_{n2} = [x_{21}, x_{22}, \cdots, x_{2n}]^T, \cdots$, $Q_{ni} = [x_{i1}, x_{i2}, \cdots, x_{in}]^T, \cdots$, $Q_{nm} = [x_{m1}, x_{m2}, \cdots, x_{mn}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points (as shown in Figure 2).

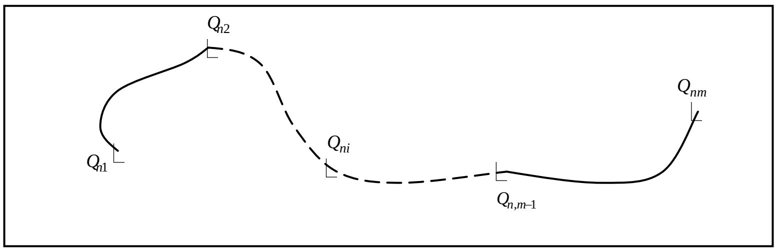

Figure 2. An example of a curve line in n-dimensional space

Any point on the curve line $Q_{ni}Q_{n,i+1} (i=1,2,\cdots,m-1)$ is then represented by

$$Q_{nir} = \begin{bmatrix} x_{i1r} \\ x_{i2r} \\ \vdots \\ x_{inr} \end{bmatrix} = \begin{bmatrix} a_{i1}r^3+b_{i1}r^2+c_{i1}r+d_{i1} \\ a_{i2}r^3+b_{i2}r^2+c_{i2}r+d_{i2} \\ \vdots \\ a_{in}r^3+b_{in}r^2+c_{in}r+d_{in} \end{bmatrix} \triangleq \begin{bmatrix} p_{i1}(r) \\ p_{i2}(r) \\ \vdots \\ p_{in}(r) \end{bmatrix} \tag{10}$$

$$\left(i=1,2,\cdots,m-1, \quad r\in[0,1]\right)$$

The true curve line of $Q_{n1}Q_{n2}\cdots Q_{ni}\cdots Q_{nm}$ can be represented by

$$\phi_{nir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \\ \vdots \\ \mu_{inr} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \\ \vdots \\ f_{in}(r) \end{bmatrix} \tag{11}$$

$$\left(i=1,2,\cdots,m-1, \quad r\in[0,1]\right)$$

5.2 The truncation error of a curve line in $n-$ dimensional space

For simplicity, it can be assumed that the cubic interpolation is piecewise cubic Hermite interpolation.

Some assumptions about the functions $f_{ij}(r)(i=1,2,\cdots,m;$ $j=1,2,\cdots,n; r\in[0,1])$ need to be made in order to be able to estimate the interpolation error.

It is assumed $f_{ij}(r)\in C^4[0,1]$, giving

$$\begin{aligned} \left|R_{ij}(r)\right| &= \left|f_{ij}(r)-p_{ij}(r)\right| \\ &\le \frac{1}{4!}M_{4ij}r^2(1-r)^2 \le \frac{1}{384}M_{4ij} \triangleq T_{ij} \end{aligned} \tag{12}$$

Where $M_{4ij}=\max\limits_{r\in[0,1]}\left|f_{ij}^{(4)}(r)\right|$.

Hence, the truncation error of a curve line can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \\ \vdots \\ T_{in} \end{bmatrix} (i=1,2,\cdots,m-1).$$

6 THE TRUNCATION ERROR FOR A CURVE LINE IN A THREE-DIMENSIONAL SPACE

6.1 Definition of a curve line in three-dimensional space

The curve line is formed by the three-dimensional vectors: $Q_{31}=[x_{11},x_{12},x_{13}]^T$, $Q_{32}=[x_{21},x_{22},x_{23}]^T,\cdots,$ $Q_{3i}=[x_{i1},x_{i2},x_{i3}]^T,\cdots,$ $Q_{3m}=[x_{m1},x_{m2},x_{m3}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points.

Any point on the curve line $Q_{3i}Q_{3,i+1}(i=1,2,\cdots,m-1)$ is then represented by

$$Q_{3ir} = \begin{bmatrix} x_{i1r} \\ x_{i2r} \\ x_{i3r} \end{bmatrix} = \begin{bmatrix} a_{i1}r^3+b_{i1}r^2+c_{i1}r+d_{i1} \\ a_{i2}r^3+b_{i2}r^2+c_{i2}r+d_{i2} \\ a_{i3}r^3+b_{i3}r^2+c_{i3}r+d_{i3} \end{bmatrix} \triangleq \begin{bmatrix} p_{i1}(r) \\ p_{i2}(r) \\ p_{i3}(r) \end{bmatrix} \tag{13}$$

$$\left(i=1,2,\cdots,m-1, \quad r\in[0,1]\right)$$

The true curve line of $Q_{31}Q_{32}\cdots Q_{3i}\cdots Q_{3m}$ can be represented by

$$\phi_{3ir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \\ \mu_{i3r} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \\ f_{i3}(r) \end{bmatrix} \left(i=1,2,\cdots,m-1, \quad r\in[0,1]\right) \tag{14}$$

6.2 The truncation error of a curve line in three-dimensional space

For simplicity, it is assumed that the cubic interpolation is piecewise cubic Hermite interpolation.

Some assumptions about the functions $f_{ij}(r)(i=1,2,\cdots,m;$ $j=1,2,3; r\in[0,1])$ need to be made in order to be able to estimate the error of interpolation.

It is assumed that $f_{ij}(r)\in C^4[0,1]$, giving

$$\begin{aligned} \left|R_{ij}(r)\right| &= \left|f_{ij}(r)-p_{ij}(r)\right| \le \frac{1}{4!}M_{4ij}r^2(1-r)^2 \\ &\le \frac{1}{384}M_{4ij} \triangleq T_{ij} \end{aligned} \tag{15}$$

Where $M_{4ij}=\max\limits_{r\in[0,1]}\left|f_{ij}^{(4)}(r)\right|$.

Hence, the curve line truncation error can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{bmatrix} (i=1,2,\cdots,m-1).$$

7 THE TRUNCATION ERROR MODEL FOR A CURVE LINE IN A TWO-DIMENSIONAL SPACE

7.1 Definition of a curve line in two-dimensional space

The curve line is formed by two-dimensional vectors: $Q_{21}=[x_{11},x_{12}]^T$, $Q_{22}=[x_{21},x_{22}]^T,\cdots,$ $Q_{2i}=[x_{i1},x_{i2}]^T,\cdots,$ $Q_{2m}=[x_{m1},x_{m2}]^T$, where $i=1,2,\cdots,m$ and m is the number of composition points.

Any point on the curve line $Q_{2i}Q_{2,i+1}(i=1,2,\cdots,m-1)$ is then represented by

$$Q_{2ir} = \begin{bmatrix} x_{i1r} \\ x_{i2r} \end{bmatrix} = \begin{bmatrix} a_{i1}r^3 + b_{i1}r^2 + c_{i1}r + d_{i1} \\ a_{i2}r^3 + b_{i2}r^2 + c_{i2}r + d_{i2} \end{bmatrix} \square \begin{bmatrix} p_{i1}(r) \\ p_{i2}(r) \end{bmatrix} \tag{16}$$

$$\left(i = 1, 2, \cdots, m-1, \quad r \in [0,1]\right)$$

The true curve line of $Q_{21}Q_{22} \cdots Q_{2i} \cdots Q_{2m}$ can be represented by

$$\phi_{2ir} = \begin{bmatrix} \mu_{i1r} \\ \mu_{i2r} \end{bmatrix} = \begin{bmatrix} f_{i1}(r) \\ f_{i2}(r) \end{bmatrix} \left(i = 1, 2, \cdots, m-1, \quad r \in [0,1]\right) \tag{17}$$

7.2 The truncation error of a curve line in two-dimensional space

For simplicity, it is assumed that the cubic interpolation is piecewise cubic Hermite interpolation.

Some assumptions about the functions $f_{ij}(r)(i = 1, 2, \cdots, m;$ $j = 1, 2; r \in [0,1])$ need to be made in order to be able to estimate the error of interpolation.

It is assumed that $f_{ij}(r) \in C^4[0,1]$, giving

$$\left|R_{ij}(r)\right| = \left|f_{ij}(r) - p_{ij}(r)\right| \le \frac{1}{4!} M_{4ij} r^2 (1-r)^2$$
$$\le \frac{1}{384} M_{4ij} \square T_{ij} \tag{18}$$

Where $M_{4ij} = \max_{r \in [0,1]} \left|f_{ij}^{(4)}(r)\right|$.

Hence, the curve line truncation error can be represented by

$$\begin{bmatrix} T_{i1} \\ T_{i2} \end{bmatrix} (i = 1, 2, \cdots, m-1).$$

8 CONCLUSIONS AND DISCUSSION

To provide a full picture about the overall error of a line, we need to quantify (a) the error of the model that is used to interpolate the line, and (b) the propagated error from the error of the component nodes of the line. This paper presented a research on estimation the error of the interpolation model of the lines, which is a step further to the early studies on propagated error of lines.

A method for estimate the model error, truncation error, of the lines have been developed in this study based on the approximation theory. A line feature in GIS can be either interpolated by linear or nonlinear functions, our research find out that of the model error of a linear interpolated line is larger than that from a nonlinear interpolation method, such as a cubic interpolation method. Such as analysis result can be used as a reference for the selection interpolation methods for lines..

By integration of the analysis result of model error in this study, and the propagated error of the lines in the early study, we can have better estimation on the overall error of lines.

A further extension of this study will be to investigate truncation error of other interpolation methods, such as hybrid interpolation method.

ACKNOWLEDGMENT

The work presented in this paper is supported by, Hong Kong Research Grants Council (Project No.: PolyU 5276/08E), The Hong Kong Polytechnic University (Project No.: G-YX0P G-YG66 G-YF24 1-ZV4F G-U 753), and the National Natural Science Foundation, China (Project No. 40629001).